Energetic lattices and braids

Joint work with Jean SERRA

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Overview

1. Optimal Cut
2. Conditions
3. Energetic Lattice
4. Braids
5. Future directions

This is part of my thesis [9].
In this talk i will present

- Energetic lattice : theorhetical basis for dynamical programming (DP)
- Braids of Partitions : Family preserving energetic ordering and DP substructure

I will not present

- Constrained optimization on hierarchies: global & local constraints [9], [18]
- Ground truth energies, net openings, ground truth fusion : [10], [11]

Problem Formulation

Input

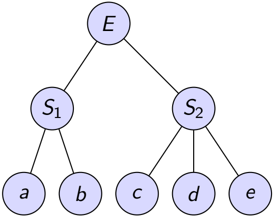
Hierarchy of Partitions

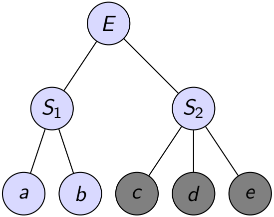
Mumford-shah
Haussdorf-distance
Total variation

Energy ω

Luminance/Chrominance
Proximity measure
Stereo disparity

Functions



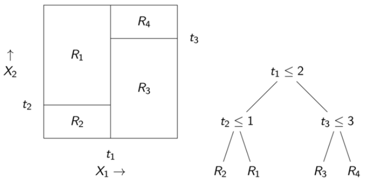


Problems :

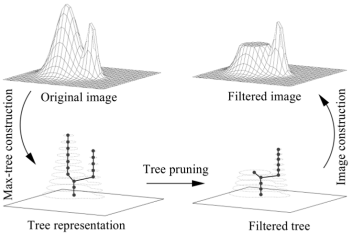
- Tractable solution ?
- Uniqueness conditions ?
- Larger Family of partitions ?

$$\underset{\pi \in \Pi(E,H)}{\text{minimize}} \sum_{S \in \pi} \omega(S)$$

Optimization on hierarchies



Decision tree cost-complexity pruning [4]



Rate-distortion minimization, level line selection. [16], [3]

Guigues Scale-sets/λ-cuts [8]

- **Given** : Hierarchy H, Energies $\omega_\phi, \omega_\partial : S \rightarrow \mathbb{R}$.
- Calculate the nested subtrees or hierarchy of partitions with increasing scale parameter λ.

Definitions

Partitions

Non-void mutually disjoint subsets of space E whose union restitutes E

$$\pi = \{S_i \subseteq E \mid \cup_i S_i = E, \ \forall i,j \ S_i \cap S_j = \emptyset\} \tag{1}$$

Partial Partitions [14]

Partition restricted to subset $S \subset E$:

$$\pi(S) = \{A_i \mid A_i \subseteq S, \ \forall i,j \ A_i \cap A_j = \emptyset\} = \pi \sqcap \{S\} \tag{2}$$

where $S = \cup_i A_i$ is called the support of $\pi(S)$.

Refinement ordering and partition lattice

Set of all partitions on space E fors a complete lattice (unique supremum/infimum) for the refinement ordering. If $\pi_i \leq \pi_j$, each class of $S_i \in \pi_i$ including a point $x \in E$ will be included in the class $S_j \in \pi_j$ including x.

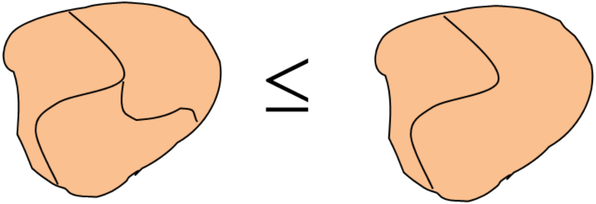


Figure : Refinement ordering.

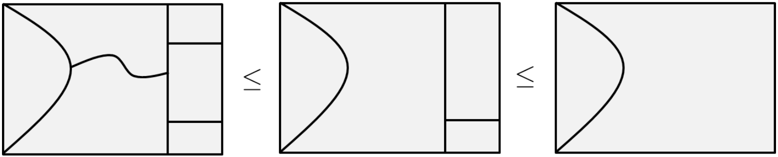
$$\boxed{\pi_i \leq \pi_j \; S_i(x) \subseteq S_j(x)} \tag{3}$$

Hierarchy of Partitions (HOP)

- An indexed family of partitions : $\{\pi_i, i \in I \subset \mathbb{Z}\}$
- A Hierarchy H is :

$$\boxed{H = \{\pi_i, i \in I\} \text{ s.t. } \forall i \leq k \implies \pi_i \leq \pi_k, I \subset \mathbb{Z}}$$

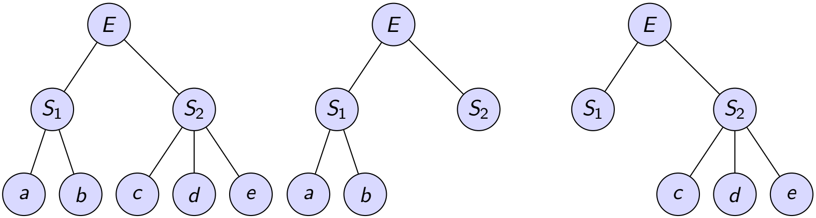
- π_0 is the finest partition in the family and is called the leaves.
- π_0 contains a finite number of leaves.



$\pi_0 \leq \pi_1 \leq \pi_2$

Elements $S \in \pi, \pi \in H$ are called the classes of the hierarchy H.

Cuts



- A Cut is a partition created with classes $S \in H$.
- It can also be seen as the set of subtrees possible from the original hierarchy/tree.4
- Examples : ($\sqcup$ is the disjoint union operator.)
  - $\pi_0 = a \sqcup b \sqcup c \sqcup d \sqcup e$
  - $\pi = a \sqcup b \sqcup S_2$
  - $\pi = S_1 \sqcup c \sqcup d \sqcup e$
- $\Pi(E, H)$: family of all cuts possible using classes from H.

Energies

- The family of partial partitions $\mathcal{D}$ or **PPs** is the set of all partial partitions possible of E.
- The energy is a value associated with each partial partition

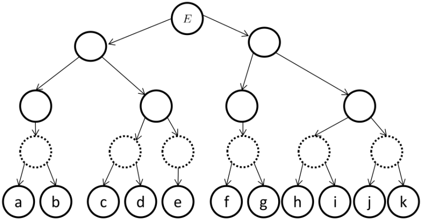
$$\boxed{\omega : \textbf{PPs} \rightarrow \mathbb{R}} \tag{4}$$

- To obtain the final energy of $\pi(S)$ a partial partition we need a composition function/law :

$$\boxed{\omega(\pi(S)) = \sum_{A_i \in \pi(S)} \omega(A_i)} \tag{5}$$

Dynamic Program

$$\pi^*(S) = \begin{cases} \{S\}, & \text{if } \omega(S) \leq \sum(\omega(\pi^*(a))), a \in \pi(S) \\ \bigsqcup_{a \in \pi(S)} \pi^*(a), & \text{otherwise} \end{cases}$$



Dynamic Program

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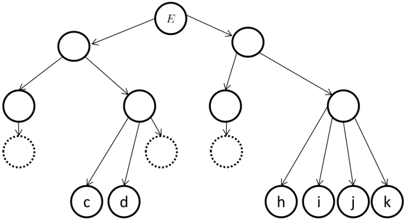
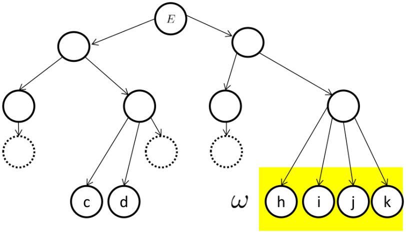


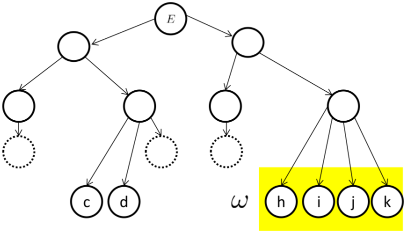
$$\pi^*(S) = \begin{cases} \{S\}, & \text{if } \omega(S) \leq \sum(\omega(\pi^*(a))), a \in \pi(S) \\ \bigsqcup_{a \in \pi(S)} \pi^*(a), & \text{otherwise} \end{cases}$$



Can we do other operations than additive ?

Dynamic Program

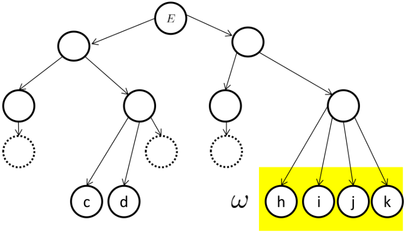
$$\pi^*(S) = \begin{cases} \{S\}, & \text{if } \omega(S) \leq \sum(\omega(\pi^*(a))), a \in \pi(S) \\ \bigsqcup_{a \in \pi(S)} \pi^*(a), & \text{otherwise} \end{cases}$$



What conditions preserve the DP ?

Dynamic Program

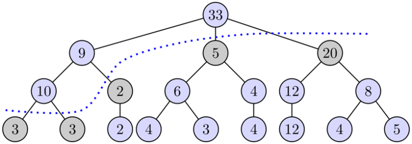
$$\pi^*(S) = \begin{cases} \{S\}, & \text{if } \omega(S) \leq \sum(\omega(\pi^*(a))), a \in \pi(S) \\ \bigsqcup_{a \in \pi(S)} \pi^*(a), & \text{otherwise} \end{cases}$$



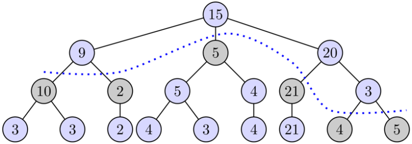
Only local comparisions performed in DP, though we claim global optimum.

Energy composition

Additive composition :



Supremum composition :



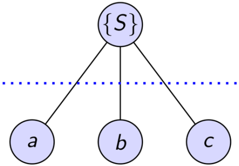
- Additive :

$$\pi^*(S) = \begin{cases} \{S\}, & \text{if } \omega(S) \leq \sum_{a \in \pi(S)} \omega(\pi^*(a)) \\ \bigsqcup_{a \in \pi(S)} \pi^*(a), & \text{otherwise} \end{cases}$$

- S more energetic than all its descendants.

$$\omega(S^*) \geq \bigvee_{\pi(S)} \omega(T_i)$$

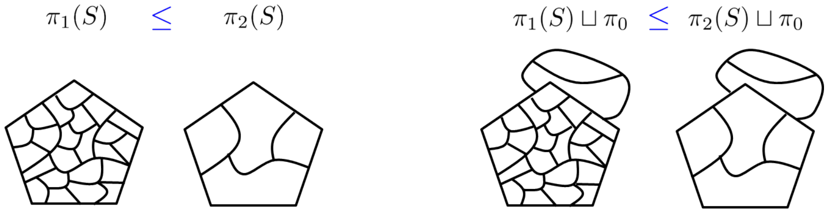
Generalized Minkowski composition



$$\omega(\pi(S), \alpha) = \left[\sum_{\text{children}} \omega(u)^{\alpha} \right]^{\frac{1}{\alpha}}$$

- **Additive**: [4],[16],[8]
- **Supremum, Dominant Ancestor**: [19],[20],[21]
- **Minkowski parameter**: [15]
- **Max-pooling type, alternating compositions**

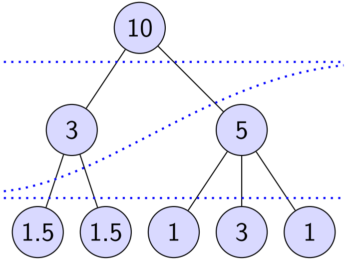
h-increasingness on HOP



$$\omega(\pi_1(S)) \leq \omega(\pi_2(S)) \implies \omega(\pi_1(S) \sqcup \pi_0) \leq \omega(\pi_2(S) \sqcup \pi_0) \tag{6}$$

h-increasingness provides the DP sub-structure necessary for optimum [17]

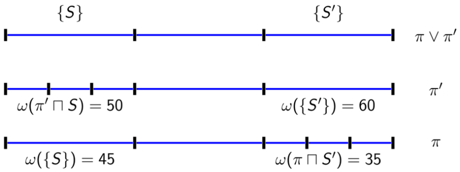
Singular eneriges



$$\boxed{\forall\ \pi(S) \in \Pi(S),\quad \omega(\{S\}) \neq \omega(\pi(S))\}}$$

Various authors indirectly use the singularity for a unique solution.

Energetic Ordering



$$\pi \preceq_\omega \pi' \;\Leftrightarrow\; \forall S \in \pi \vee \pi' \text{ we have } \omega(\pi \sqcap \{S\}) \leq \omega(\pi' \sqcap \{S\})$$

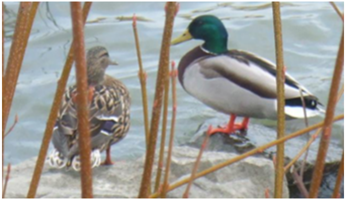
- One never evaluates energy of a partition during the dynamic program but only of partial partitions.
- The energetic lattice ($\preceq_\omega, \vee_\omega$) derives from the energetic order.
- Existence of unique solution when ω singular.
- local minimum $\implies$ global minimum.
- Given H, ω we have an energetic lattice iff ω is singular.
  h-increasingess

Segmentation Example

luminance fidelity term

$\omega_\varphi(T) = \int_T ||I(x) - \mu(T)||^2 \, \mathrm{d}x$

Initial image



chrominance fidelity term

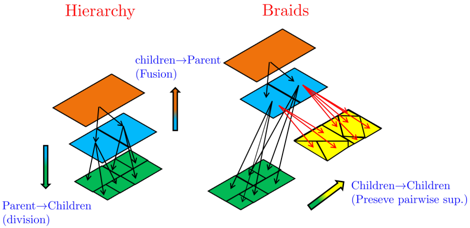
$\omega_\varphi(T) = \sum_i \int_T ||c_i(x) - \mu_i(T)||^2 \, \mathrm{d}x$

$$\boxed{\omega(\pi(S), \lambda) = \sum_k \omega_\varphi(T_k) + \lambda \omega_\partial(T_k)}$$

- Contour length : $\omega_\partial(T_k) = \partial T_k$
- λ was fixed to have same coding cost

Pair-wise Refinement supremum are hierarchical [12]

$$\boxed{\forall\ \pi_1, \pi_2 \in B \quad \Rightarrow \quad \pi_1 \vee \pi_2 \in \Pi(H, E) \setminus \{E\}}$$

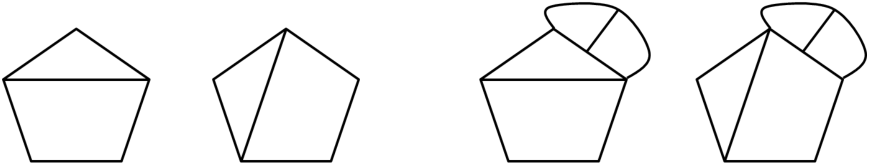


All properties of optimiszation of the hierarchies extend to braids:

- h-increasingness
- dynamic programing
- energetic lattice

h-increasingness for Braids

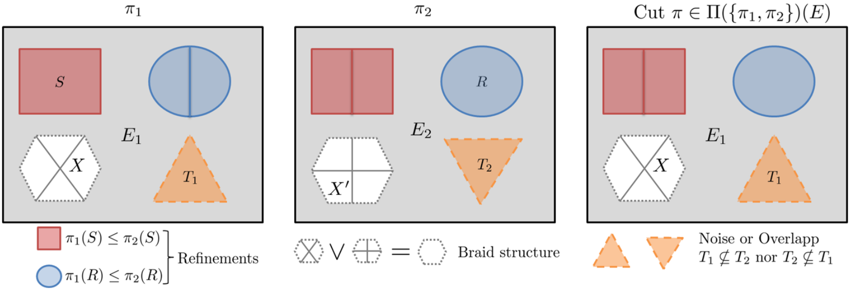
$\pi_1(S) \not\leq \pi_2(S)$ $\qquad$ $\pi_1(S) \sqcup \pi_0 \not\leq \pi_2(S) \sqcup \pi_0$



$$\omega(\pi_1(S)) \leq \omega(\pi_2(S)) \quad \Rightarrow \quad \omega(\pi_1(S) \sqcup \pi_0) \leq \omega(\pi_2(S) \sqcup \pi_0)$$

Why Braids ?

- Uncertain boundaries $\implies$ mutliple partial partitions are optimal.
- Multivariate energy minimization, partial partitions across components.
- Accomodates variablity in Human & Machine segmenations.
- Better DP infimum & compatible with Energetic Lattice.



Future work

- Given a hierarchy of partitions that are totally not ordered by refinement, i.e. $H = \{\pi_i, i \in I\}$ s.t. $\forall i, j, k$, either $\pi_i \leq \pi_j \leq \pi_k$, $\pi_i \geq \pi_j \leq \pi_k$, $\pi_i \leq \pi_j \geq \pi_k$, $\pi_i \geq \pi_j \geq \pi_k$. How do formulate the DP to achieve the global optima ? This is possible in algorithms that perform local refinements, but always working from a fixed partition lattice.
- Understanding the optimal cut and pruning problems in ensembles. Recent paper on Cost-complexity pruning of Random Forests explores this subject. [13]
- Maximally weighted independent set as segmentation follows a similar dynamic program [5]
- Project under study : Multi-class graph-cuts to optimize energies using the α-extension algorithm [7].

Thank you!
On to Hyperspectral imaging

References I

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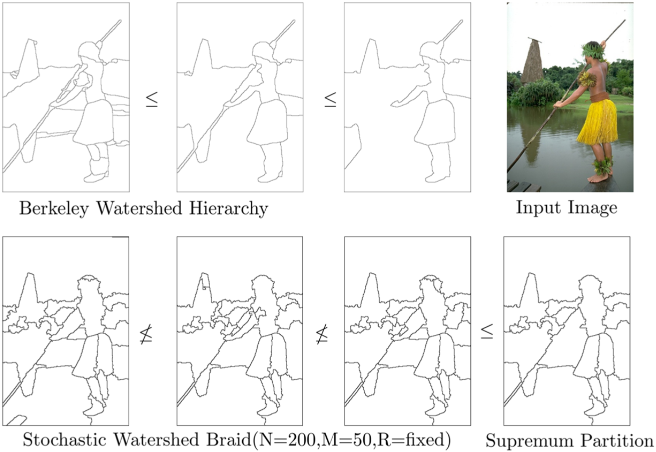
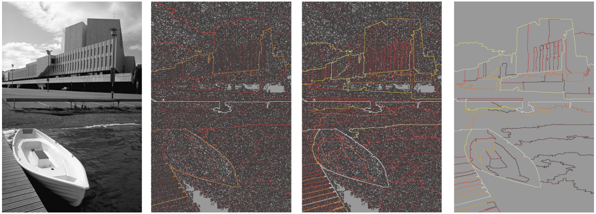
Braid Examples

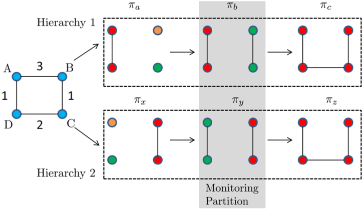


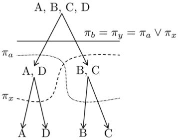
- UCM Hierarchy : [2]
- Stochastic Watershed : [1]

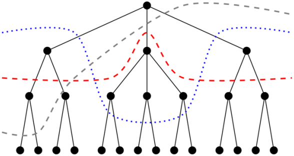
Braid Examples



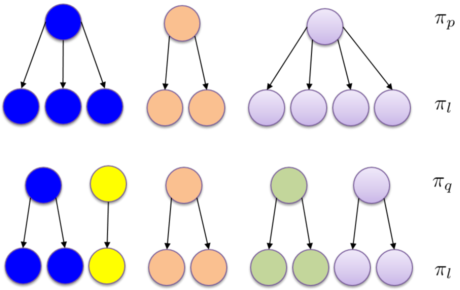
Input image   Area & Vol. watersheds [6]   Monitor Hierarchy







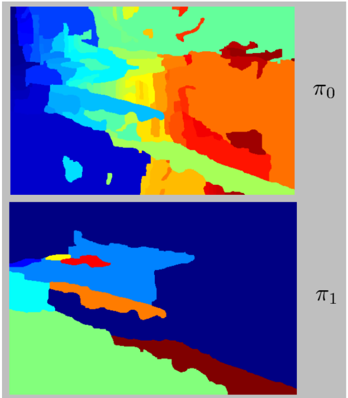
Minimum Spanning Tree (MST) Braid, $B_5 = \{\pi \in \Pi(E) \mid \|\pi\| = 5\}$



For $\pi_i \in H$
    For $S \in \pi_i$
        if($\omega(\pi_q(S) < \pi_p^*(S))$
            $\pi^*(S) = \pi_q(S)$
        else
            recompose new children

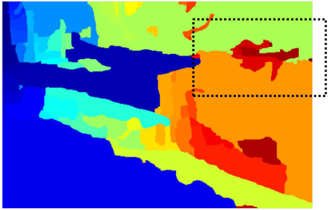
For every class of parent partition π_p we regroup children in π_l at random to search possible finer parent level π_q

Example

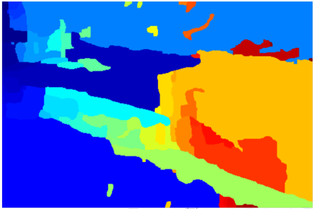


2-level Hierarchy

Optimal cut from hierarchy $\pi^*(\lambda = 550)$
84 classes $\omega(\pi^*(\lambda = 550)) = 20313749.3404$

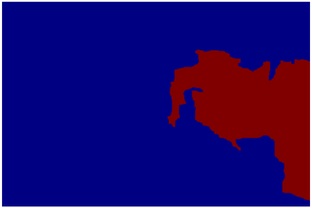


(Perturbed Hierarchy forms a braid)
Opitmal Cut from Braid $\pi_p^*(\lambda = 550)$
74 classes $\omega(\pi_p^*(\lambda = 550)) = 19672933.8262$

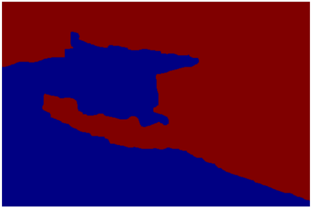


Example

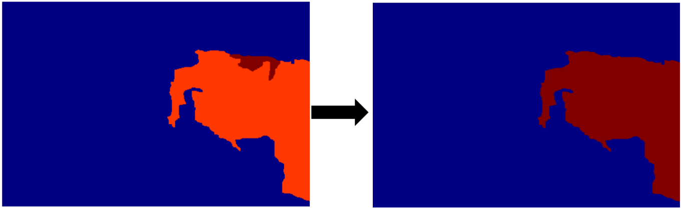
Class in partition $\pi_0 \in H$



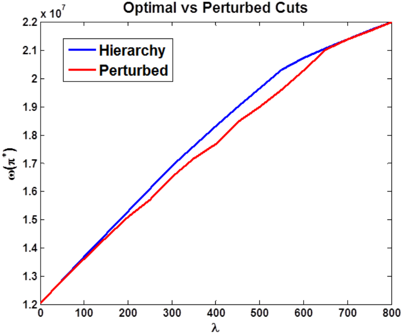
Class in partition $\pi_1 \in H$



Two classes chosen at random during perturbation step
not grouped into single parent in H
Produces better partial partition and lower energy



Example



Observation

- Perturbation and search is λ-dependent.
- Locally random perturbations. (More evoled versions possible)
- Refinement order respected.