Overview

1. Optimal Cut
2. Conditions
3. Energetic Lattice
4. Braids
5. Future directions

This is part of my thesis [9].
In this talk i will present

- Energetic lattice : theoretical basis for dynamical programming (DP)
- Braids of Partitions : Family preserving energetic ordering and DP substructure

I will not present

- Constrained optimization on hierarchies: global & local constraints [9], [18]
- Ground truth energies, net openings, ground truth fusion: [10], [11]
**Problem Formulation**

**Input**

- **Hierarchy of Partitions**
- Mumford-shah Haussdorf-distance Total variation
- Luminance/Chrominance Proximity measure Stereo disparity

**Energy** $\omega$

**Problems:**

- Tractable solution?
- Uniqueness conditions?
- Larger Family of partitions?

**minimize**

$$\sum_{S \in \pi} \omega(S)$$

$$\pi \in \Pi(E,H)$$
Optimization on hierarchies

Decision tree cost-complexity pruning [4]

Rate-distortion minimization, level line selection. [16], [3]

Guigues Scale-sets/λ-cuts [8]

- **Given**: Hierarchy \( H \), Energies \( \omega_{\phi}, \omega_{\partial} : S \rightarrow \mathbb{R} \).
- Calculate the nested subtrees or hierarchy of partitions with increasing scale parameter \( \lambda \).
Definitions

Partitions

Non-void mutually disjoint subsets of space $E$ whose union restitutes $E$

\[ \pi = \{ S_i \subseteq E \mid \bigcup_i S_i = E, \ \forall i, j \ S_i \cap S_j = \emptyset \} \] (1)

Partial Partitions [14]

Partition restricted to subset $S \subset E$:

\[ \pi(S) = \{ A_i \mid A_i \subseteq S, \ \forall i, j \ A_i \cap A_j = \emptyset \} = \pi \cap \{ S \} \] (2)

where $S = \bigcup_i A_i$ is called the support of $\pi(S)$. 
Refinement ordering and partition lattice

Set of all partitions on space $E$ for a complete lattice (unique supremum/infimum) for the refinement ordering. If $\pi_i \leq \pi_j$, each class of $S_i \in \pi_i$ including a point $x \in E$ will be included in the class $S_j \in \pi_j$ including $x$.

\[ \pi_i \leq \pi_j \quad S_i(x) \subseteq S_j(x) \] (3)

Figure : Refinement ordering.
Hierarchy of Partitions (HOP)

- An indexed family of partitions: \( \{ \pi_i, i \in I \subseteq \mathbb{Z} \} \)
- A Hierarchy \( H \) is:

\[
H = \{ \pi_i, i \in I \} \quad \text{s.t.} \quad \forall i \leq k \implies \pi_i \leq \pi_k, I \subseteq \mathbb{Z}
\]

- \( \pi_0 \) is the finest partition in the family and is called the leaves.
- \( \pi_0 \) contains a finite number of leaves.

Elements \( S \in \pi, \pi \in H \) are called the classes of the hierarchy \( H \).
A **Cut** is a partition created with classes $S \in H$.

It can also be seen as the set of subtrees possible from the original hierarchy/tree.

Examples: ($\sqcup$ is the disjoint union operator.)

- $\pi_0 = a \sqcup b \sqcup c \sqcup d \sqcup e$
- $\pi = a \sqcup b \sqcup S_2$
- $\pi = S_1 \sqcup c \sqcup d \sqcup e$

$\Pi(E, H)$: family of all cuts possible using classes from $H$. 
Energies

- The family of partial partitions $\mathcal{D}$ or $\text{PPs}$ is the set of all partial partitions possible of $E$.
- The energy is a value associated with each partial partition

$$\omega : \text{PPs} \to \mathbb{R}$$

(4)

- To obtain the final energy of $\pi(S)$ a partial partition we need a composition function/law:

$$\omega(\pi(S)) = \sum_{A_i \in \pi(S)} \omega(A_i)$$

(5)
Dynamic Program

\[ \pi^*(S) = \begin{cases} \{S\}, & \text{if } \omega(S) \leq \sum(\omega(\pi^*(a))), a \in \pi(S) \\ \bigcup_{a \in \pi(S)} \pi^*(a), & \text{otherwise} \end{cases} \]
\[ \pi^*(S) = \begin{cases} 
\{ S \}, & \text{if } \omega(S) \leq \sum (\omega(\pi^*(a))), a \in \pi(S) \\
\bigcup_{a \in \pi(S)} \pi^*(a), & \text{otherwise} 
\end{cases} \]
Dynamic Program

\[ \pi^*(S) = \begin{cases} 
\{S\}, & \text{if } \omega(S) \leq \sum(\omega(\pi^*(a))), \ a \in \pi(S) \\
\bigcup_{a \in \pi(S)} \pi^*(a), & \text{otherwise} 
\end{cases} \]

Can we do other operations than additive?
Dynamic Program

\[ \pi^*(S) = \begin{cases} 
\{S\}, & \text{if } \omega(S) \leq \sum_{a \in \pi(S)} \omega(\pi^*(a)), a \in \pi(S) \\
\bigcup_{a \in \pi(S)} \pi^*(a), & \text{otherwise} 
\end{cases} \]

What conditions preserve the DP?
Dynamic Program

\[
\pi^*(S) = \begin{cases} 
\{S\}, & \text{if } \omega(S) \leq \sum(\omega(\pi^*(a))), a \in \pi(S) \\
\bigcup_{a \in \pi(S)} \pi^*(a), & \text{otherwise}
\end{cases}
\]

Only local comparisons performed in DP, though we claim global optimum.
Energy composition

Additive composition:

\[ \pi^*(S) = \begin{cases} 
\{S\}, & \text{if } \omega(S) \leq \sum_{a \in \pi(S)} \omega(\pi^*(a)) \\
\bigcup_{a \in \pi(S)} \pi^*(a), & \text{otherwise} 
\end{cases} \]

- Additive:

- \( S \) more energetic than all its descendants.

Supremum composition:

\[ \omega(S^*) \geq \bigvee_{\pi(S)} \omega(T_i) \]

Kiran BANGALORE RAVI (CRISTaL)

Energetic lattices and braids

March 22, 2017 11 / 21
Generalized Minkowski composition

\[ \omega(\pi(S), \alpha) = \left[ \sum_{\text{children}} \omega(u)^\alpha \right]^{\frac{1}{\alpha}} \]

- **Additive**: [4],[16],[8]
- **Supremum, Dominant Ancestor**: [19],[20],[21]
- **Minkowski parameter**: [15]
- **Max-pooling type, alternating compositions**
$h$-increasingness provides the DP sub-structure necessary for optimum [17]
Various authors indirectly use the singularity for a unique solution.
Energetic Ordering

One never evaluates energy of a partition during the dynamic program but only of partial partitions.

The energetic lattice \((\leq, \lor)\) derives from the energetic order.

Existence of unique solution when \(\omega\) singular.

Local minimum \(\implies\) global minimum.

Given \(H, \omega\) we have an energetic lattice iff \(\omega\) is singular.

\(h\)-increasingess
**Segmentation Example**

**luminance fidelity term**

\[ \omega_\varphi(T) = \int_T ||l(x) - \mu(T)||^2 \, dx \]

**chrominance fidelity term**

\[ \omega_\varphi(T) = \sum_i \int_T ||c_i(x) - \mu_i(T)||^2 \, dx \]

**Contour length**

\[ \omega(\pi(S), \lambda) = \sum_k \omega_\varphi(T_k) + \lambda \omega_{\partial}(T_k) \]

- Contour length: \( \omega_{\partial}(T_k) = \partial T_k \)
- \( \lambda \) was fixed to have same coding cost
Pair-wise Refinement supremum are hierarchical [12]

\[ \forall \pi_1, \pi_2 \in B \implies \pi_1 \lor \pi_2 \in \Pi(H, E) \setminus \{E\} \]

All properties of optimization of the hierarchies extend to braids:
- h-increasingness
- dynamic programing
- energetic lattice
$h$-increasingness for Braids

\[
\pi_1(S) \not\leq \pi_2(S) \quad \text{and} \quad \pi_1(S) \sqcup \pi_0 \not\leq \pi_2(S) \sqcup \pi_0
\]

\[
\omega(\pi_1(S)) \leq \omega(\pi_2(S)) \implies \omega(\pi_1(S) \sqcup \pi_0) \leq \omega(\pi_2(S) \sqcup \pi_0)
\]
Why Braids?

- Uncertain boundaries $\implies$ multiple partial partitions are optimal.
- Multivariate energy minimization, partial partitions across components.
- Accomodates variability in Human & Machine segmentations.
- Better DP infimum & compatible with Energetic Lattice.

\[ \begin{align*}
\pi_1(S) \leq & \pi_2(S) \quad \text{Refinements} \\
\pi_1(R) \leq & \pi_2(R) \\
\end{align*} \]

Cut $\pi \in \Pi(\{\pi_1, \pi_2\})(E)$

\[ \begin{align*}
\bigvee \quad \bigoplus = \bigcirc \quad \text{Braid structure} \\
\end{align*} \]

Noise or Overlap $T_1 \notin T_2$ nor $T_2 \notin T_1$
Future work

- Given a hierarchy of partitions that are totally not ordered by refinement, i.e. $H = \{\pi_i, i \in I\}$ s.t. $\forall i, j, k$, either $\pi_i \leq \pi_j \leq \pi_k$, $\pi_i \geq \pi_j \leq \pi_k$, $\pi_i \leq \pi_j \geq \pi_k$, $\pi_i \geq \pi_j \geq \pi_k$. How do formulate the DP to achieve the global optima? This is possible in algorithms that perform local refinements, but always working from a fixed partition lattice.

- Understanding the optimal cut and pruning problems in ensembles. Recent paper on Cost-complexity pruning of Random Forests explores this subject. [13]

- Maximally weighted independent set as segmentation follows a similar dynamic program [5]

- Project under study: Multi-class graph-cuts to optimize energies using the $\alpha$-extension algorithm [7].
Thank you!
On to Hyperspectral imaging
References


Braid Examples

- UCM Hierarchy: [2]
- Stochastic Watershed: [1]
Braid Examples


Minimum Spanning Tree (MST) Braid,  \( B_5 = \{ \pi \in \Pi(E) \mid \|\pi\| = 5 \} \)
Perturbing Hierarchies: Searching for better optimum

Random re-compositions on hierarchies

For $\pi_i \in H$

For $S \in \pi_i$

if($\omega(\pi_q(S) < \pi^*_p(S))$

$\pi^*(S) = \pi_q(S)$

else

recompose new children

For every class of parent partition $\pi_p$ we regroup children in $\pi_l$ at random to search possible finer parent level $\pi_q$
Optimal cut from hierarchy $\pi^*(\lambda = 550)$
84 classes $\omega(\pi^*(\lambda = 550)) = 20313749.3404$

(Perturbed Hierarchy forms a braid)
Optimal Cut from Braid $\pi^*_p(\lambda = 550)$
74 classes $\omega(\pi^*_p(\lambda = 550)) = 19672933.8262$
Example

Class in partition $\pi_0 \in H$

Class in partition $\pi_1 \in H$

Two classes chosen at random during perturbation step not grouped into single parent in $H$
Produces better partial partition and lower energy
**Example**

**Observation**
- Perturbation and search is $\lambda$-dependent.
- Locally random perturbations. (More evolved versions possible)
- Refinement order respected.