Hierarchies and climbing energies

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Abstract. A new approach is proposed for finding the "best cut" in a hierarchy of partitions by energy minimization. Said energy must be "climbing" i.e. it must be hierarchically and scale increasing. It encompasses separable energies [5], [9] and those which composed under supremum [14], [12]. It opens the door to multivariate data processing by providing laws of combination by extrema and by products of composition.

1 Introduction

A hierarchy of image transforms, or of image operators, intuitively is a series of progressive simplified versions of the said image. This hierarchical sequence is also called a pyramid. In the particular case that we take up here, the image transforms will always consist in segmentations, and lead to increasing *partitions* of the space. Now, a multi-scale image description can rarely be considered as an end in itself. It often requires to be completed by some operation that summarizes the hierarchy into the "best cut" in a given sense. Two questions arise then, namely:

1. Given a hierarchy H of partitions and an energy ω on its partial partitions, how to combine classes of this hierarchy for obtaining a new partition that minimizes ω?
2. When ω depends on integer j, i.e. $\omega = \omega^j$, how to generate a sequence of minimum partitions that increase with j, which therefore should form a minimum hierarchy?

These questions have been taken up by several authors. The present work pursues, indeed, the method initiated by Ph. Salembier and L. Garrido for generating thumbnails [9], well formalized for additive energies by L.Guigues et al [5], [5] and extended by J. Serra in [10]. In [9], the superlative "best", in "best cut", is interpreted as the most accurate image simplification for a given compression rate. We take up this Lagrangian approach again in the example of section below. In [5], the "best" cut requires linearity and affinity assumptions. However, one can wonder whether these two hypotheses are the very cause of the properties found by the authors. Indeed, for solving problem 1 above, the alternative and simpler condition of hierarchical increasingness is proposed in [10], and is shown to encompass optimizations which are neither linear nor

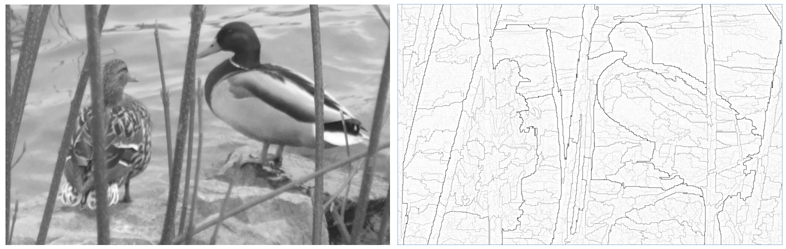

Fig. 1. Left: Initial image, Right: Saliency map of the hierarchy H obtained from image.

affine, such as P. Soille's constraint connectivity [12], or Zanoguerra's lasso based segmentations [14].

Our study is related to the ideas developed by P. Arbelaez et al [1] in learning strategies for segmentation. It is also related to the approach of J. Cardelino et al [3] where Mumford and Shah functional is modified by the introduction of shape descriptors. Similarly C. Ballester et al. [2] use shape descriptors to yield compact representations.

The present paper aims to solve the above questions, 1 and 2. The former was partly treated in [10], where the concept of h-increasingness was introduced as a sufficient condition. More deeply, it is proved in [10] that an energy satisfies the two minimizations of questions 1 and 2 if and only if it is climbing. The present paper summarizes without proofs the major results of the technical report [10], yet unpublished. The results of [10] are briefly reminded in section 2; the next section introduces the climbing energies (definition 3) and states the main result of the text (theorem 2); the last section, number 4, develops an example.

2 Hierarchical increasingness (reminder)

The space under study (Euclidean, digital, or else) is denoted by E and the set of subsets of E by $P(E)$. A partition $\pi(S)$ associated with a set $S \in \mathcal{P}(E)$ is called *partial partition* of E of support S [8]. The family of all partial partitions of set E is denoted by $\mathcal{D}(E)$, or simply by $\mathcal{D}$. A hierarchy H is a finite chain of partitions π_i, i.e.

$$H = \{\pi_i, 0 \leq i \leq n \mid i \leq k \leq n \Rightarrow \pi_i \leq \pi_k\}, \tag{1}$$

where π_n is the partition $\{E\}$ of E in a single class.

The partitions of a hierarchy may be represented by their classes, or by the saliency map of the edges[6],[4], as depicted in Figure 1, or again by a family tree where each node of bifurcation is a class S, as depicted in Figure 2. The classes of π_{i-1} at level $i-1$ which are included in class S_i are said to be *the sons* of S_i.

Denote by $\mathcal{S}(H)$ the set of all classes S of all partitions involved in H. Clearly, the descendants of each S form in turn a hierarchy $H(S)$ of summit S, which is included in the complete hierarchy $H = H(E)$.

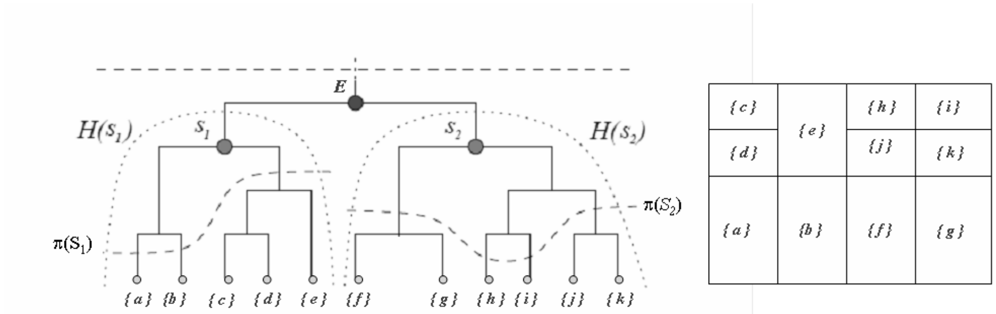

Fig. 2. Left, hierarchical tree; right, the corresponding space structure. S_1 and S_2 are the nodes sons of E, and $H(S_1)$ and $H(S_1)$ are the associated sub-hierarchies. π_1 and π_2 are cuts of $H(S_1)$ and $H(S_1)$ respectively, and $\pi_1 \sqcup \pi_2$ is a cut of E.

2.1 Cuts in a hierarchy

Any partition π of E whose classes are taken in $\mathcal{S}$ defines a *cut* in hierarchy H. The set of all cuts of E is denoted by $\Pi(E) = \Pi$. Every "horizontal" section $\pi_i(H)$ at level i is obviously a cut, but several levels can cooperate in a same cut, such as $\pi(S_1)$ and $\pi(S_2)$, drawn with thick dotted lines in Figure 2. Similarly, the partition $\pi(S_1) \sqcup \pi(S_2)$ generates a cut of $H(E)$. The symbol $\sqcup$ is used here for expressing that groups of classes are concatenated. Each class S may be in turn the root of sub-hierarchy $H(S)$ where S is the summit, and in which (partial) cuts may be defined. whose it is the summit. Let $\Pi(S)$ be the family of all cuts of $H(S)$. The union of all these cuts, when node S spans hierarchy H is denoted by

$$\widetilde{\Pi}(H) = \cup\{\Pi(S), S \in \mathcal{S}(H)\}. \tag{2}$$

2.2 Cuts of minimum energy and *h*-increasingness

Definition 1. *An energy $\omega : \mathcal{D}(E) \to \mathbb{R}^+$ is a non negative numerical function over the family $\mathcal{D}(E)$ of all partial partitions of set E. An optimum cut $\pi^* \in \Pi(E)$ of E, is one that minimizes ω, i.e. $\omega(\pi^*) = \inf\{\omega(\pi) \mid \pi \in \Pi(E)\}$.*

The problem of unicity of optimum cut is not treated here (refer [11]).

Definition 2. *[10] Let π_1 and π_2 be two partial partitions of same support, and π_0 be a partial partition disjoint from π_1 and π_2. An energy ω on $\mathcal{D}(E)$ is said to be* hierarchically increasing, or *h*-increasing, *in $\mathcal{D}(E)$ when, $\pi_0, \pi_1, \pi_2 \in \mathcal{D}(E)$, π_0 disjoint of π_1 and π_2, we have*

$$\omega(\pi_1) \leq \omega(\pi_2) \quad \Rightarrow \quad \omega(\pi_1 \sqcup \pi_0) \leq \ \omega(\pi_2 \sqcup \pi_0). \tag{3}$$

Implication (3) is illustrated in Figure 3. When the partial partitions are embedded in a hierarchy H, then Rel.(3) allows us an easy characterization of

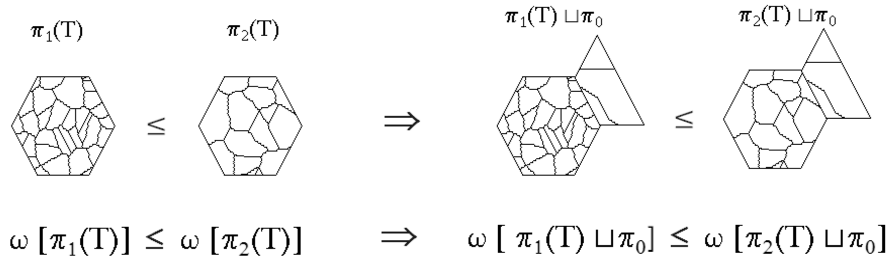

Fig. 3. Hierachical increasingness.

the cuts of minimum energy of H, according to the following property, valid for the class $\mathcal{H}$ of all finite hierarchies on E.

Theorem 1. *Let $H \in \mathcal{H}$ be a finite hierarchy, and ω be an energy on $\mathcal{D}(E)$. Consider a node S of H with p sons $T_1..T_p$ of optimum cuts $\pi_1^*, ..\pi_p^*$. The cut of optimum energy of summit S is, in a non exclusive manner, either the cut*

$$\pi_1^* \sqcup \pi_2^* .. \sqcup \pi_p^*, \tag{4}$$

or the partition of S into a unique class, if and only if S is h-increasing (proof given in [11])

The condition of h-increasingness (3) opens into a broad range of energies, and is easy to check. It encompasses that of Mumford and Shah, the separable energies of Guigues [5] [9], as well as energies composed by suprema [12] [14], and many other ones [11]. Moreover, any weighted sum $\Sigma \lambda_j \omega^j$ of *h-increasing* energies with positive λ_j is still *h-increasing* energies, as well as, under some conditions, any supremum and infimum of *h-increasing* energies [11]. The condition (3) yields a dynamic algorithm, due to Guigues, for finding the optimum cut $\pi^*(H)$ in one pass [5].

2.3 Generation of *h*-increasing energies

The energy $\omega : \mathcal{D}(E) \to \mathbb{R}^+$ has to be defined on the family $\mathcal{D}(E)$ of all partial partitions of E. An easy way to obtain a h-increasing energy consists in taking, firstly, an arbitrary energy ω on all sets $S \in \mathcal{P}(E)$, considered as one class partial partitions $\{S\}$, and then in extending ω to all partial partitions by some law of composition. The h-increasingness is introduced here by the law of composition, and not by $\omega[\mathcal{P}(E)]$. The first laws which come to mind are, of course, addition, supremum, and infimum, and indeed we can state:

Proposition 1. *Let E be a set and $\omega : \mathcal{P}(E) \to \mathbb{R}^+$ an arbitrary energy defined on $\mathcal{P}(E)$, and let $\pi \in \mathcal{D}(E)$ be a partial partition of classes $\{S_i, 1 \leq i \leq n\}$. Then the three extensions of ω to the partial partitions $\mathcal{D}(E)$*

$$\omega(\pi) = \bigvee_i \omega(S_i), \quad \omega(\pi) = \bigwedge_i \omega(S_i), \quad \textit{and} \;\; \omega(\pi) = \textstyle\sum_i \omega(S_i), \tag{5}$$

are h-increasing energies.

A number of other laws are compatible with h-increasingness. One could use the product of energies, the difference sup-inf, the quadratic sum, and their combinations. Moreover, one can make depend ω on more than one class, on the proximity of the edges, on another hierarchy, etc..

3 Climbing energies

The usual energies are often given by finite sequences $\{\omega^j, 1 \leq j \leq p\}$ that depend on a positive index, or parameter, j. Therefore, the processing of hierarchy H results in a sequence of p optimum cuts π^{j*}, of labels $1 \leq j \leq p$. *A priori*, the π^{j*} are not ordered, but if they were, i.e. if

$$j \leq k \quad \Rightarrow \quad \pi^{j*} \leq \pi^{k*}, \qquad j,k \in J, \tag{6}$$

then we should obtain a nice progressive simplification of the optimum cuts. For getting it, we need to combine h-increasingness with the supplementary axiom (7) of *scale increasingness*, which results in the following *climbing energies.*

Definition 3. *We call* climbing energy *any family $\{\omega^j, 1 \leq j \leq p\}$ of energies over $\widetilde{\Pi}$ which satisfies the three following axioms, valid for $\omega^j, 1 \leq j \leq p$ and for all $\pi \in \Pi(S)$, $S \in \mathcal{S}$*

- *i) each ω^j is* h-increasing,
- *ii) each ω^j admits a* single optimum cutting,
- *iii) the $\{\omega^j\}$ are* scale increasingness, *i.e. for $j \leq k$, each support $S \in \mathcal{S}$ and each partition $\pi \in \Pi(S)$, we have that*

$$j \leq k \ \text{ and } \omega^j(S) \leq \omega^j(\pi) \Rightarrow \omega^k(S) \leq \omega^k(\pi), \qquad \pi \in \Pi(S), \quad S \in \mathcal{S}. \tag{7}$$

Axiom i) and ii) allow us to compare the same energy at two different levels, whereas iii) compares two different energies at the same level. The relation (7) means that, as j increases, the ω^j's preserve the sense of energetic differences between the nodes of hierarchy H and their partial partitions. In particular, all energies of the type $\omega^j = j\omega$ are scale increasing.

The climbing energies satisfy the very nice property to order the optimum cuts with respect to the parameter j:

Theorem 2. *Let $\{\omega^j,\ 1 \leq j \leq p\}$ be a family of energies, and let π^{j*} (resp. π^{k*}) be the optimum cut of hierarchy H according to the energy ω^j (resp. ω^k). The family $\{\pi^{j*}, 1 \leq j \leq p\}$ of the optimum cuts generates a unique hierarchy H^* of partitions, i.e.*

$$j \leq k \quad \Rightarrow \quad \pi^{j*} \leq \pi^{k*}, \qquad 1 \leq j \leq k \leq p \tag{8}$$

if and only if the family $\{\omega^j\}$ is a climbing energy (proof given in [11]).

Such a family is climbing in two senses: for each j the energy climbs pyramid H up to its best cut (h-increasingness), and as j varies, it generates a new pyramid to be climbed (scale-increasingness). Relation (8) has been established by L. Guigues in his Phd thesis [5] for affine and separable energies, called by him multiscale energies. However, the core of the assumption (7) concerns the propagation of energy through the scales $(1...p)$, rather than affinity or linearity, and allows non additive laws. In addition, the set of axioms of the climbing energies 3 leads to an implementation simpler than that of [5].

4 Examples

We now present two examples of energies composed by rule of supremum and another by addition. In all cases, the energies depend on a scalar parameter k such that the three families $\{\omega^k\}$ are climbing. The reader may find several particular climbing energies in the examples treated in [5],[14],[13],and [9].

4.1 Increasing binary energies

The simplest energies are the binary ones, which take values 1 and 0 only. We firstly observe that the relation $\pi \sqsubseteq \pi_1$, where $\pi_1 = \pi \sqcup \pi'$ is made of the classes of π plus other ones, is an ordering. A binary energy ω such that for all $\pi, \pi_0, \pi_1, \pi_2 \in \mathcal{D}(E)$

$$\omega \text{ is } \sqsubseteq\text{-increasing, i.e. } \omega(\pi) = 1 \quad \Rightarrow \quad \omega(\pi \sqcup \pi_0) = 1$$

$$\omega(\pi_1) = \omega(\pi_2) = 0 \quad \Rightarrow \quad \omega(\pi_1 \sqcup \pi_0) = \omega(\pi_2 \sqcup \pi_0),$$

is obviously h-increasing, and conversely. Here are two examples of this type.

Large classes removal One wants to suppress the very small classes, considered as noise, and also the largest ones, considered as not significant. Associate with each $S \in \mathcal{P}(E)$ the energy $\omega^k(\langle S \rangle) = 0$ when $area(S) \leq k$, and $\omega^k(\langle S \rangle) = 1$ when not, and compose them by sum, $\pi = \sqcup \langle S_i \rangle \Rightarrow \omega^k(\pi) = \sum_i \omega^k(\langle S_i \rangle)$. Therefore the energy of a partition equals the number of its classes whose areas are larger than k. Then the class of the optimum cut at point $x \in E$ is the larger class of the hierarchy that contains x and has an area not greater than k.

Soille-Grazzini minimization [13],[12] A numerical function f is now associated with hierarchy H. Consider the range of variation $\delta(S) = \max\{f(x), x \in S\} - \min\{f(x), x \in S\}$ of f inside set S, and the h-increasing binary energy $\omega^k(\langle S \rangle) = 0$ when $\delta(S) \leq k$, and $\omega^k(\langle S \rangle) = 1$ when not. Compose ω according the law of the supremum, i.e. $\pi = \sqcup \langle S_i \rangle \Rightarrow \omega^k(\pi) = \bigvee_i \omega^k(\langle S_i \rangle)$. Then the class of the optimum cut at point $x \in E$ is the larger class of H whose range of variation is $\leq j$. When the energy ω^k of a father equals that of its sons, one keeps the father when $\omega^k = 0$, and the sons when not.

4.2 Additive energies under constraint

The example of additive energy that we now develop is a variant of the creation of thumbnails by Ph. Salembier and L. Garrido [9]. We aim to generate "the best" simplified version of a colour image f, of components (r, g, b), when the compression rate is imposed equal to 20. The bit depth of f is 24 and the size of f is = 600x480 pixels. A hierarchy H has been obtained by previous segmentations of the luminance $l = (r+g+b)/3$ based on [4]. In each class S of H, the reduction consists in replacing the function f by its colour mean $m(S)$. The quality of this approximation is estimated by the L_2 norm, i.e.

$$\omega_\mu(S) = \sum_{x \in S} \parallel l(x) - m(S) \parallel^2 . \tag{9}$$

The coding cost for a frontier element is $\simeq 2$, which gives, for the whole S

$$\omega_\partial(S) = 24 + \mid \partial S \mid \tag{10}$$

with 24 bits for $m(S)$. We want to minimize $\omega_\mu(S)$, while preserving the cost. According to Lagrange formalism, the total energy of class S is thus written $\omega(S) = \omega_\mu(S) + \lambda^j \omega_\partial(S)$. Classically one reaches the minimum under constraint $\omega(S)$ by means of a system of partial derivatives. Now remarkably our approach replaces the of computation of derivatives by a climbing. Indeed we can access the energy a cut π by summing up that of its classes, which leads to $\omega(\pi) = \lambda^j \omega_\mu(\pi) + \omega_\partial(\pi)$. The cost $\omega_\partial(\pi)$ decreases as λ^j increases, therefore we can climb the pyramid of the best cuts and stop when $\omega_\partial(\pi) \simeq n/20$. It results in Figure 4 (left), where we see the female duck is not nicely simplified.

However, there is no particular reason to choose the same luminance l for generating the pyramid, and later as the quantity to involve in the quality estimate (9). In the RGB space, a colour vector $\overrightarrow{x}$ (r, g, b) can be decomposed in its two orthogonal projections on the grey axis, namely $\overrightarrow{l}$ of components $(l/3, l/3, l/3)$, and on the chromatic plane orthogonal to the grey axis at the origin, namely $\overrightarrow{c}$ of components $(3/\sqrt{2})(2r - g - b, 2g - b - r, 2b - r - g)$. We have $\overrightarrow{x} = \overrightarrow{l} + \overrightarrow{c}$. Let us repeat the optimization by replacing the luminance $l(x)$ in (9) by the module $|\overrightarrow{c}(x)|$ of the chrominance in x. We now find for best cut the segmentation depicted in Figure 4, where, for the same compression rate, the animals are correctly rendered, but the river background is more simplified than previously.

5 Conclusion

This paper has introduced the new concept of increasing energies. It allows to find best cuts in hierarchies of partitions, encompasses the known optimizations of such hierarchies and opens the way to combinations of energies by supremum, by infimum, and by scalar product of Lagrangian constraints. This work was funded by Agence Nationale de la Recherche through contract ANR-2010-BLAN-0205-03 KIDIKO.

Fig. 4. Left: Best cut of Duck image by optimizing by Luminance, Right: and by Chrominance.

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